Decision Problems for Linear Recurrence Sequences

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(joint work with James Worrell)

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Termination of Linear Programs

$\mathbf{x} := \mathbf{a}$;
while $\mathbf{u} \cdot \mathbf{x} \geq 0$ do
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Instance: $\langle\, \mathbf{a};\, \mathbf{u};\, \mathbf{M} \,\rangle$ over $\mathbb{Z}$ or $\mathbb{Q}$
Question: Does this program terminate?

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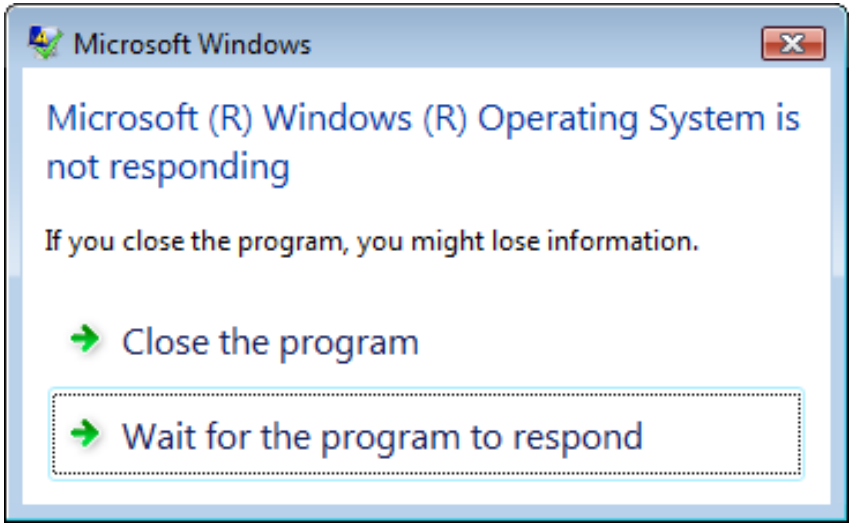
Much work on this and related problems in the literature over the last three decades:

- Manna, Pnueli, Kannan, Lipton, Sagiv, Podelski, Rybalchenko, Cook, Dershowitz, Tiwari, Braverman, Kovács, Ben-Amram, Genaim, ...
- Approaches include:
  - linear ranking functions
  - size-change termination methods
  - spectral techniques
  - ...
- Tools include:

TERMINATOR

proof tools for termination and liveness


Microsoft Windows
Microsoft (R) Windows (R) Operating System is not responding
If you close the program, you might lose information.
Close the program
Wait for the program to respond

How Software Hangs

```
    do {
        p = r = b + (2 * PTHRESH);
        if (r >= t) p = r = t;      /* too short to care about */
        else {
            while (((cmp(aTHX_ *(p-1), *p) > 0) == sense) &&
                   ((p -= 2) > q)) {}
            if (p <= q) {
                /* b through r is a (long) run.
                ** Extend it as far as possible. */
                p = q = r;
                while (((p += 2) < t) &&
                       ((cmp(aTHX_ *(p-1), *p) > 0) == sense)) q = p;
                r = p = q + 2;      /* no simple pairs, no after-run */
            }
        }
        if (q > b) {                /* run of greater than 2 at b */
            gptr *savep = p;
            p = q += 2;
            /* pick up singleton, if possible */
            if ((p == t) &&
                ((t + 1) == last) &&
                ((cmp(aTHX_ *(p-1), *p) > 0) == sense))
                savep = r = p = q = last;
            p2 = NEXT(p2) = p2 + (p - b); ++runs;
            if (sense)
                while (b < --p) {
                    const gptr c = *b;
                    *b++ = *p;
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                }
            p = savep;
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        while (q < p) {             /* simple pairs */
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while $\mathbf{u} \cdot \mathbf{x} \geq 0$ do

$\mathbf{x} := \mathbf{M}\mathbf{x} + \mathbf{b}$;

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If $\mathbf{M}$ *is diagonalisable and has dimension* 9×9 *or less, Termination is decidable.*

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Markov Chain Problem

Instance: $\langle$ stochastic matrix $\mathbf{M}$; $r \in (0, 1]$ $\rangle$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1, 0, \ldots, 0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq r$?

Positivity and Zeros of Linear Recurrence Sequences

$u_0 = 0,\ u_1 = 1$

$u_{n+2} = u_{n+1} + u_n$

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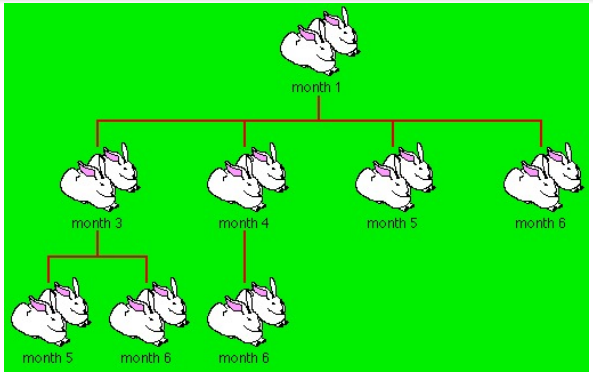

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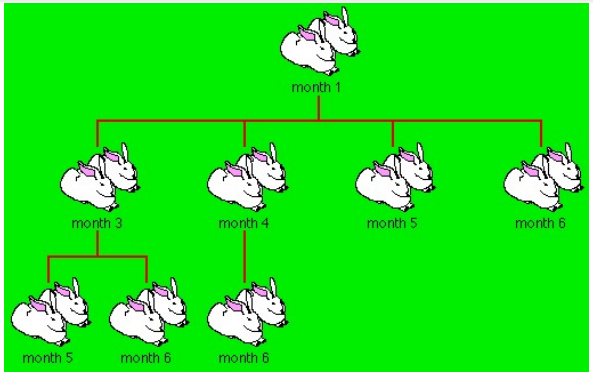

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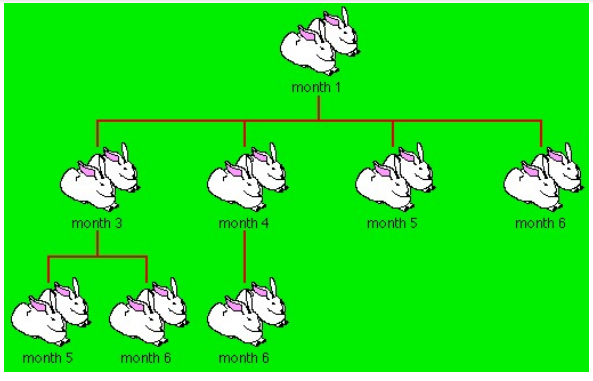
$u_{n+5} = u_{n+4} + u_{n+3} - \frac{1}{3}u_n$



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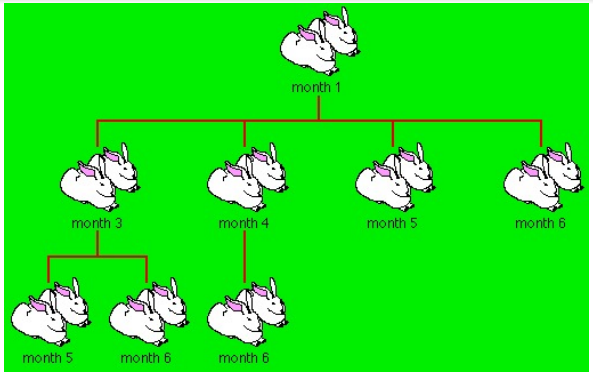

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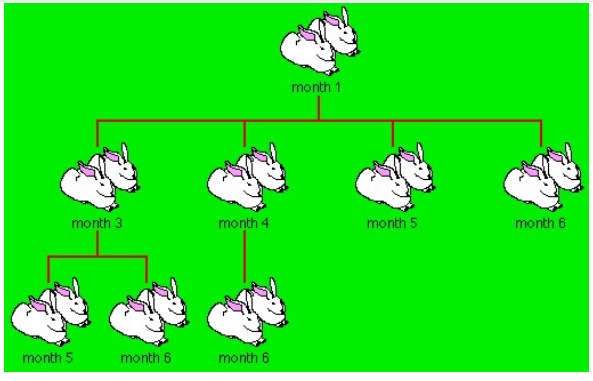

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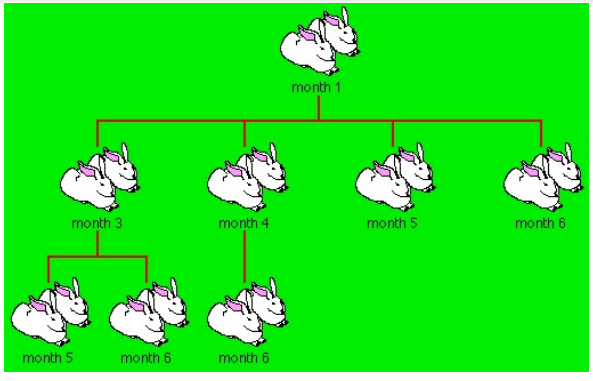

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Question: Is it the case that $\forall n,\ u_n \geq 0$?

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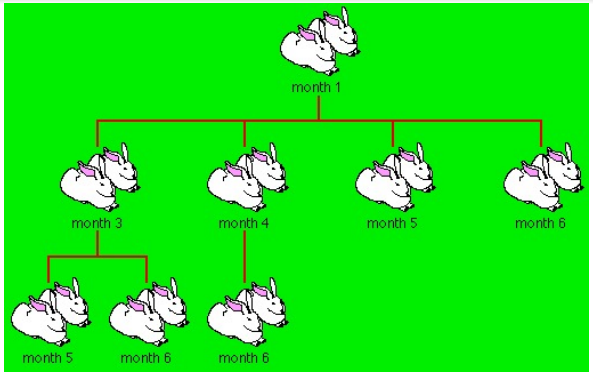

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Skolem Problem

Instance: A linear recurrence sequence $\langle u_n \rangle$
Question: Does $\exists n$ such that $u_n = 0$?

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Fibonacci: A Closer Look

$u_0 = 0,\ u_1 = 1$
$u_{n+2} = u_{n+1} + u_n$

- The Fibonacci sequence has **order** 2
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Linear Recurrence Sequences

- Numbers $\langle u_0, u_1, u_2, \ldots \rangle$ form a **linear recurrence sequence** if there exist k and constants $a_1, \ldots, a_k$, such that

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  In general $\lambda_1, \ldots, \lambda_k$ and all coefficients of $p_1(x), \ldots, p_m(x)$ are algebraic numbers
- If the linear recurrence sequence is **simple** then the polynomials $p_1(x), \ldots, p_m(x)$ are all **constant**

Decision Problems for Linear Recurrence Sequences

- Let $\langle u_n \rangle$ be a linear recurrence sequence

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

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Ultimate Positivity Problem

Does $\exists T$ such that, $\forall n \geq T,\ u_n \geq 0$?

Related Work and Applications

- Theoretical biology
  - Analysis of L-systems
  - Population dynamics
- Software verification
  - Termination of linear programs
- Probabilistic model checking
  - Reachability, invariance, and approximation in Markov chains
  - Stochastic logics
- Quantum computing
  - Threshold problems for quantum automata
- Economics
- Combinatorics
- Discrete linear dynamical systems
- Statistical physics
- ...

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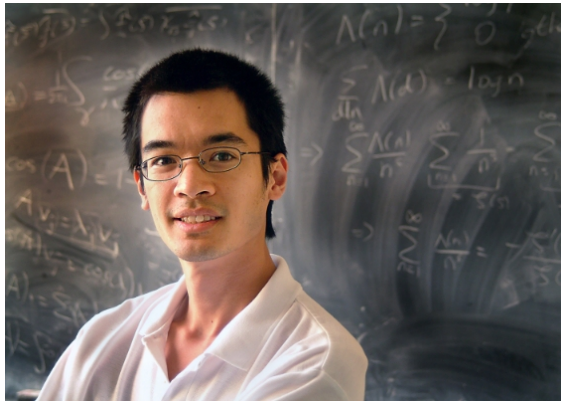

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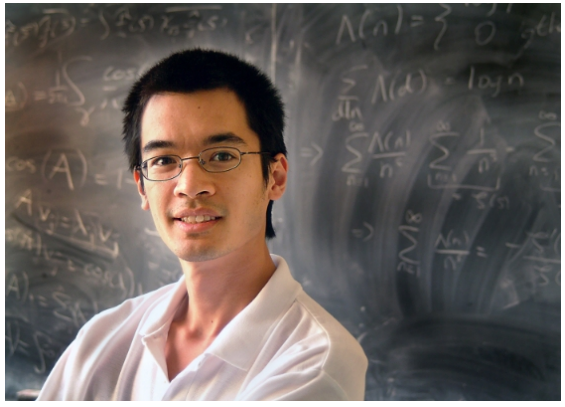

"...a mathematical embarrassment..."

Richard Lipton

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semilinear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

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- All known proofs make essential use of p-adic techniques

Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.

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Critical ingredient is Baker's theorem for linear forms in logarithms, which earned Baker the Fields Medal in 1970.

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Decidability for order 5 was announced in 2005 by four Finnish mathematicians in a technical report (as yet unpublished). Their proof appears to have a serious gap.

The Positivity and Ultimate Positivity Problems

- Positivity and Ultimate Positivity open since at least 1970s

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Decidability of Positivity $\Rightarrow$ *decidability of Skolem.*

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In *Colloquium Mathematicum* 128:1 (2012), Tangsupphathawat, Punnim, and Laohakosol claimed decidability of Positivity and Ultimate Positivity for order 4 (and noted being stuck for order 5). Unfortunately, their proof contains a major error.

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Positivity is decidable for order 5 or less.

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Theorem

At order 6, for both Positivity and Ultimate Positivity, proof of decidability would entail major breakthroughs in analytic number theory (Diophantine approximation of transcendental numbers).

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For **simple** *linear recurrence sequences of order 9 or less, Positivity is decidable.*

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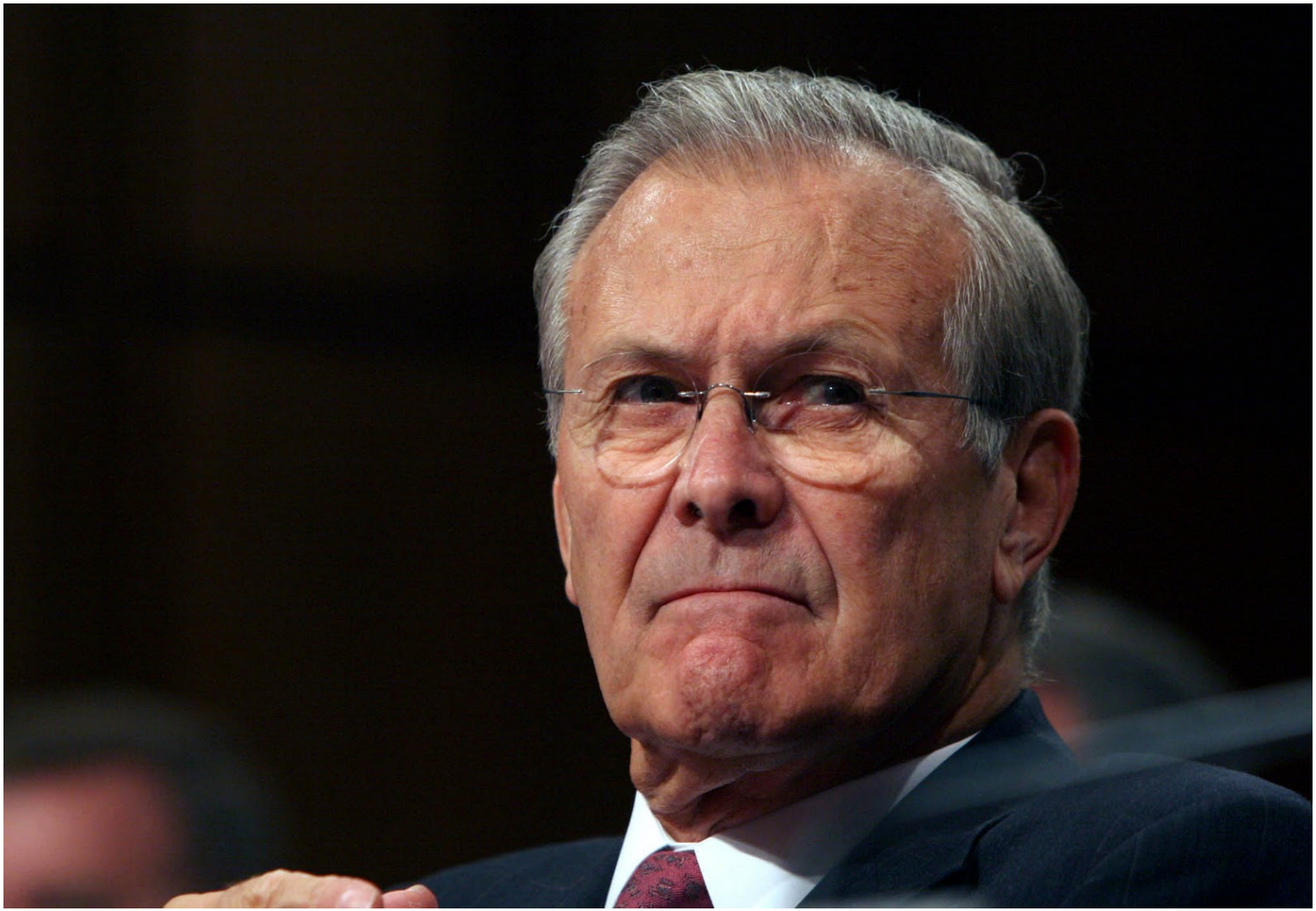
- *For each fixed order k, complexity is in P (but depends on k).*
- *In the general case, complexity is in PSPACE and co$\exists\mathbb{R}$-hard.*

Known Unknowns



"There are things that we know we don't know. . . "

Donald Rumsfeld

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Moreover, $\frac{1}{\sqrt{5}}$ is the best possible constant that will work for all real numbers x.

Diophantine Approximation

Definition

Let $x \in \mathbb{R}$. The **Lagrange constant** $L_\infty(x)$ is:

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Almost nothing else is known about any specific irrational number!

Hardness

Let $\mathcal{T} = \{\theta \in (0,1) \ : \ e^{2\pi i\theta} \in \mathbb{Q}(i)\} \ \setminus \ \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$

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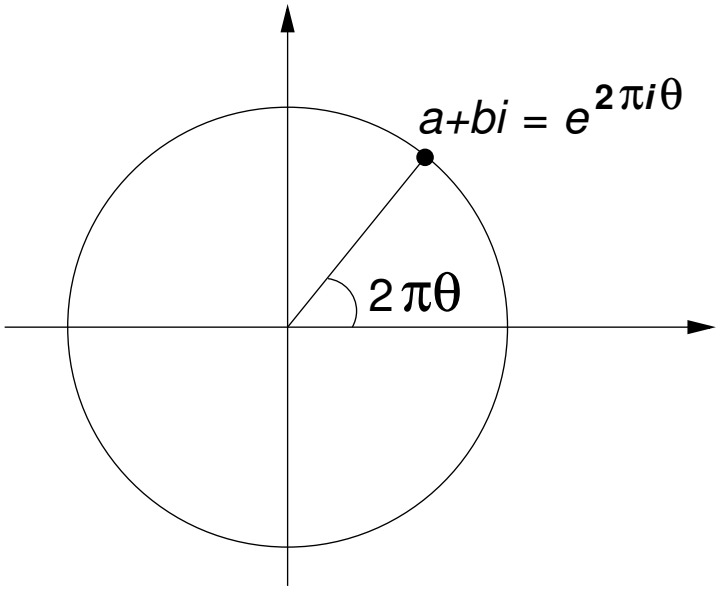

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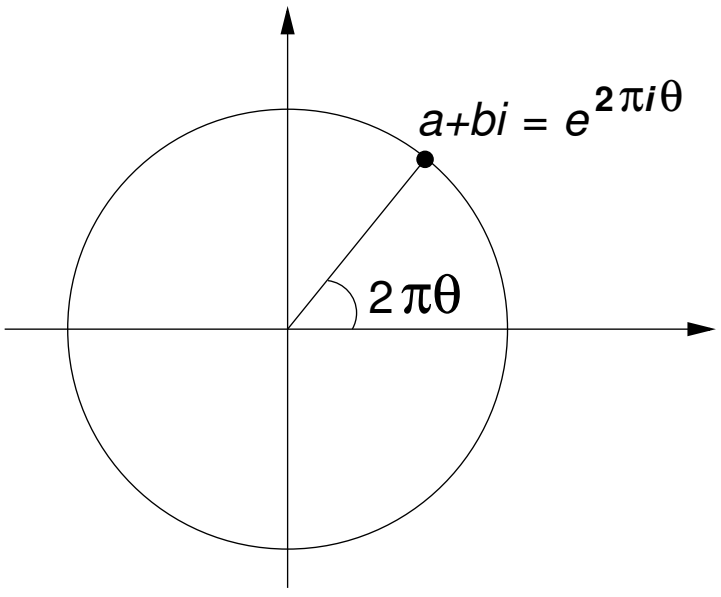

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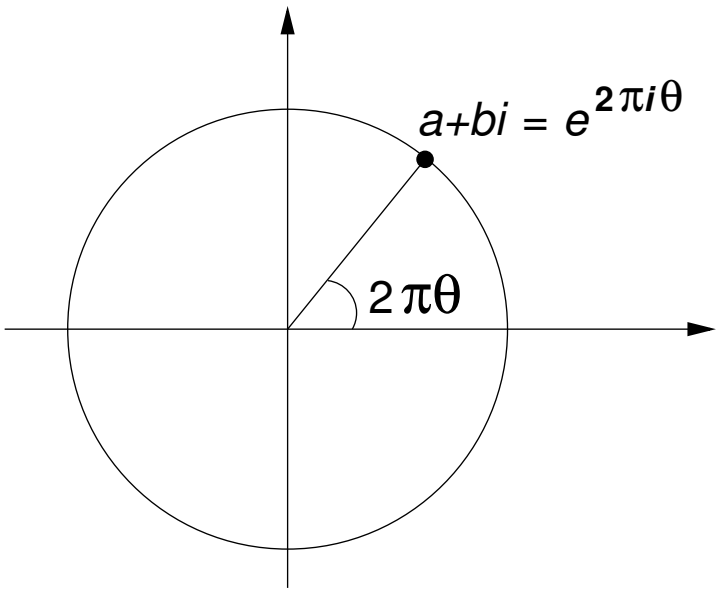

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Theorem

Suppose that Ultimate Positivity is decidable for integer linear recurrence sequences of order 6. Then for any $\theta \in \mathcal{T}$, $L_\infty(\theta)$ is computable.

Positivity of Simple LRS: Algorithm Sketch

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For simple linear recurrence sequences:

- *Ultimate Positivity is decidable for all orders.*
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2. Compute a threshold T such that $\langle u_n \rangle_{n=T}^{\infty}$ is positive.

Positivity of Simple LRS: Algorithm Sketch

Theorem

For simple linear recurrence sequences:

- *Ultimate Positivity is decidable for all orders.*
- *Positivity is decidable for orders 9 or less.*

Input: a simple linear recurrence sequence $\langle u_n \rangle_{n=0}^{\infty}$ of order ≤ 9

1. Decide if $\langle u_n \rangle_{n=0}^{\infty}$ is ultimately positive.
   If it isn't, $\langle u_n \rangle_{n=0}^{\infty}$ is **not** positive. Otherwise:
2. Compute a threshold T such that $\langle u_n \rangle_{n=T}^{\infty}$ is positive.
3. Check individually whether $u_0 \geq 0$, $u_1 \geq 0$, $\ldots$, $u_{T-1} \geq 0$.

Lower Bounds in Diophantine Approximation

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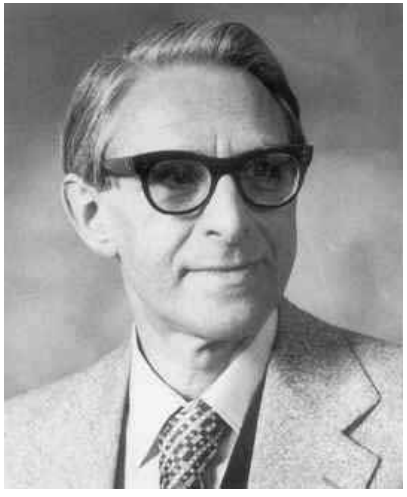

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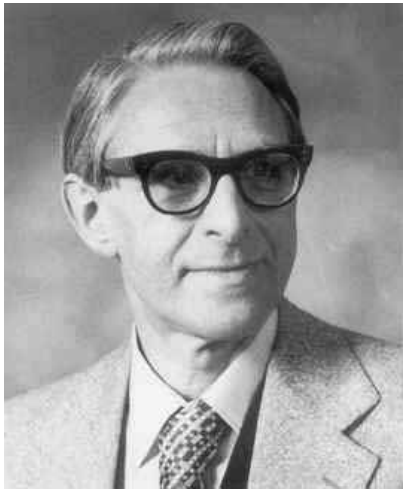

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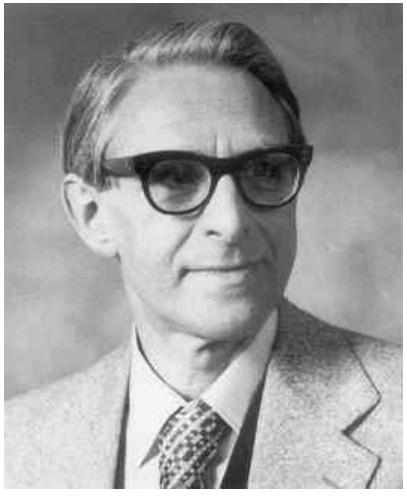

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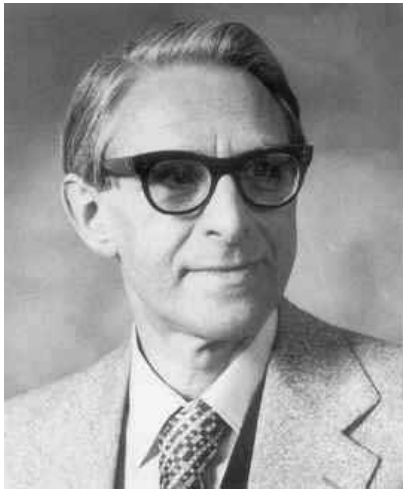

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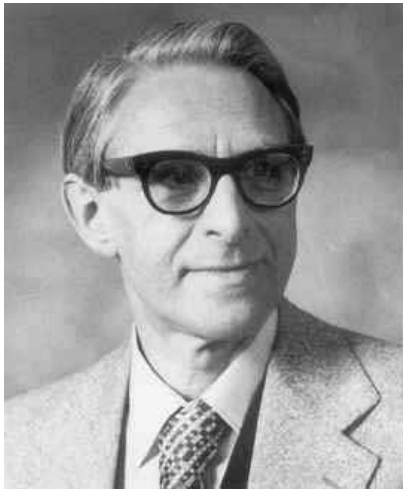

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- $\Rightarrow$ Evertse, van der Poorten, and Schlickewei's **lower bounds on sums of S-units** (1984–1985)

Lower Bounds on Sums of S-Units: A Simple Example

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- Constructive proof requires Baker's Theorem (!)

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- **No constructive proof is known !**

Lower Bounds on Sums of S-Units and Simple Linear Recurrence Sequences

We use complex algebraic-integer extensions of such results to study expressions of the form:

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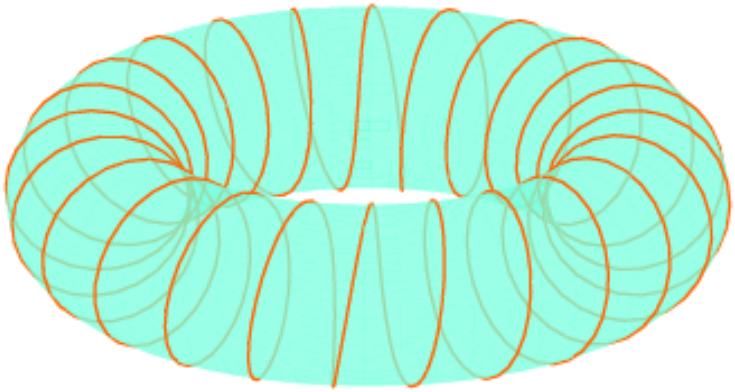

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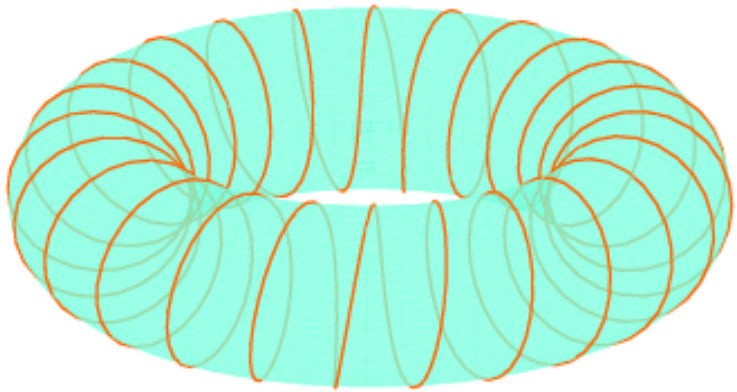
$$f(z_1, z_2, \ldots, z_k) = c_1z_1 + c_2z_2 + \ldots + c_kz_k$$



Main Tools and Techniques

- Algebraic and analytic number theory, Diophantine geometry
  - p-adic techniques
  - Baker's theorem on linear forms in logarithms
  - Kronecker's theorem on simultaneous Diophantine approximation
  - Masser's results on multiplicative relationships among algebraic numbers
  - Schmidt's Subspace theorem and Schlickewei's p-adic extension
  - Sums of S-units techniques
  - Gelfond-Schneider theorem
  - Other Diophantine geometry and approximation techniques
- Real algebraic geometry

Decision and Synthesis Problems for Linear Dynamical Systems

- A fresh look at an old area
- Lots of cool problems
- Lots of interesting mathematics
- Many connections to variety of other fields
- Funded by ERC Consolidator Grant 2015–2020
  - $\Rightarrow$ Several PhD and Postdoc positions available — please get in touch if interested!