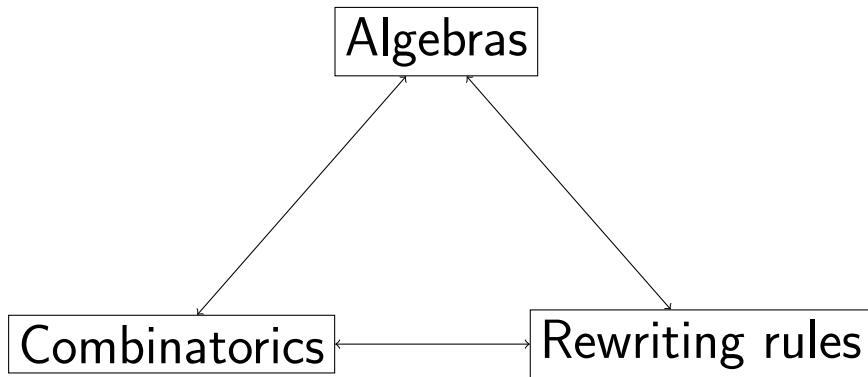


Combinatorics, rewriting rules and algebras

V. Vong

Laboratoire d'Informatique Gaspard-Monge
Université Paris-Est Marne-la-Vallée

2nd february, 2015



Definition

\mathcal{P} -algebra : $\mathcal{A} = (V, \mathcal{P}, R)$ where

- $V = \bigoplus V_n$ is a graded vector space,
- \mathcal{P} is a finite set of bilinear maps over V ,
- R is a set of relations based on elements of \mathcal{P} .

Hilbert series of \mathcal{A} :

$$\mathcal{H}_{\mathcal{A}}(t) = \sum_{n \geq 0} \dim(V_n) t^n.$$

Some algebraic problems

- 1 Compute the hilbert series of a free \mathcal{P} -algebra over one generator of degree one.
- 2 Prove the freeness of a given \mathcal{P} -algebra.

Dendriform algebras

Definition (Loday 1998)

A *dendriform algebra* $(\mathcal{A}, \succ, \prec)$:

- $\mathcal{A} : \mathbb{K}$ vector space,
- \succ and \prec : bilinear maps,
- relations :

$$\left\{ \begin{array}{l} (a \prec b) \prec c = a \prec (b \prec c + b \succ c) = a \prec (b \star c) \\ a \succ (b \succ c) = (a \star b) \succ c \\ (a \succ b) \prec c = a \succ (b \prec c) \end{array} \right.$$

Dendriform algebras

Definition (Loday 1998)

A *dendriform algebra* $(\mathcal{A}, \succ, \prec)$:

- $\mathcal{A} : \mathbb{K}$ vector space,
- \succ and \prec : bilinear maps,
- relations :

$$\begin{cases} (a \prec b) \prec c = a \prec (b \prec c + b \succ c) = a \prec (b \star c) \\ a \succ (b \succ c) = (a \star b) \succ c \\ (a \succ b) \prec c = a \succ (b \prec c) \end{cases}$$

Another way to write the relation :

$$(a \prec b) \prec c = a \prec (b \star c),$$

→

$$(\prec) \prec = \prec (\star).$$

Dendriform algebras

Definition (Loday 1998)

A *dendriform algebra* $(\mathcal{A}, \succ, \prec)$:

- \mathcal{A} : \mathbb{K} vector space,
- \succ and \prec : bilinear maps,
- relations :

$$\left\{ \begin{array}{l} (a \prec b) \prec c = a \prec (b \prec c + b \succ c) = a \prec (b \star c) \\ a \succ (b \prec c) = (a \star b) \succ c \\ (a \succ b) \prec c = a \succ (b \prec c) \end{array} \right.$$

The syntactic tree representations :

$$\begin{array}{c} \prec \\ / \quad \backslash \\ \prec \quad c \end{array} = \begin{array}{c} \prec \\ / \quad \backslash \\ \prec \quad \star \end{array} = \begin{array}{c} \prec \\ / \quad \backslash \\ \prec \quad \prec \end{array} + \begin{array}{c} \prec \\ / \quad \backslash \\ \prec \quad \succ \end{array} .$$

Dendriform algebras

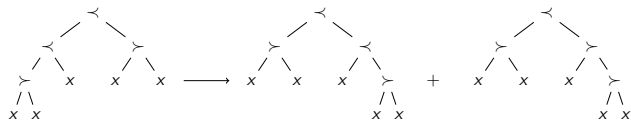
\mathcal{A} , free dendriform algebra over one generator of degree one represented by complete binary trees whose internal nodes are labeled by $\{\prec, \succ\}$ with the relations :

$$\begin{array}{c} \prec \\ / \quad \backslash \\ \prec \quad \prec \end{array} = \begin{array}{c} \prec \\ \backslash \\ \star \end{array} = \begin{array}{c} \prec \\ \backslash \\ \prec \end{array} + \begin{array}{c} \prec \\ \backslash \\ \prec \end{array} .$$

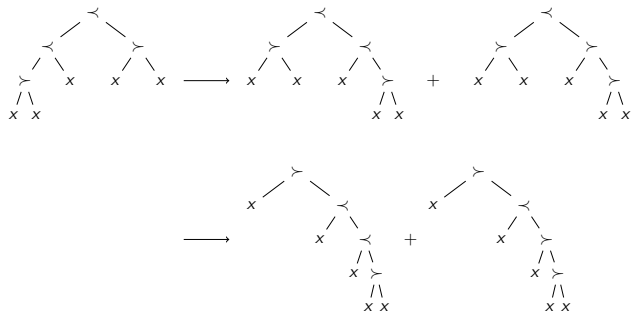
$$\begin{array}{c} \succ \\ \backslash \\ \succ \end{array} = \begin{array}{c} \succ \\ / \\ \star \end{array} = \begin{array}{c} \succ \\ / \\ \succ \end{array} + \begin{array}{c} \succ \\ / \\ \succ \end{array} .$$

$$\begin{array}{c} \prec \\ / \\ \succ \end{array} = \begin{array}{c} \succ \\ \backslash \\ \prec \end{array} .$$

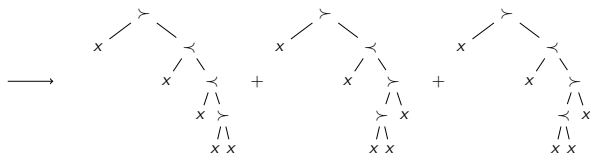
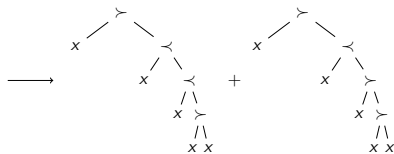
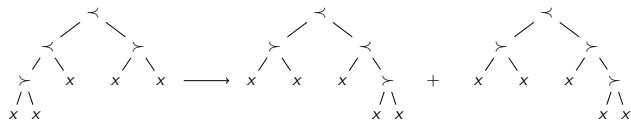
Dendriform algebras



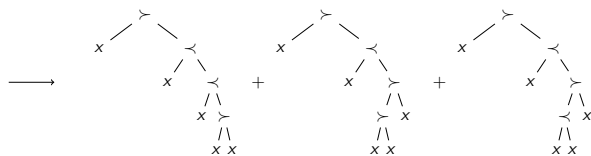
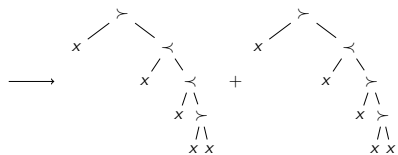
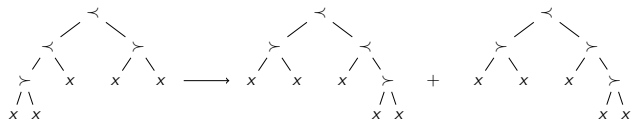
Dendriform algebras

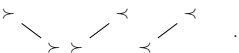


Dendriform algebras



Dendriform algebras

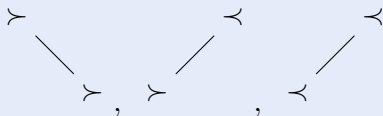


Trees avoiding  .

Dendriform algebras

Fact

The family \mathcal{F} of complete binary trees labelled by $\{\succ, \prec\}$ and avoiding



spans the free dendriform algebra over one generator.

Corollary

$$\dim(A_n) \leq C_n.$$

Definition (Malvenuto-Retenauer 1995)

$$\mathbf{FQSym} := \bigoplus_{n \geq 0} \mathbb{K}[\mathfrak{S}_n].$$

In the basis \mathbf{F} , the product is given by the shifted shuffle :

$$\mathbf{F}_\sigma \mathbf{F}_\tau = \sum_{s \in \sigma \sqcup \tau} \mathbf{F}_s$$

Dendriform algebras

Definition

$$v = v_1 \cdots v_m, w = w_1 \cdots w_n$$

- $v \sqcup w = v$ if $n = 0$,
- $v \sqcup w = w$ if $m = 0$,
- $v \sqcup w = (w \sqcup v_1 \cdots v_{m-1})v_m + (w_1 \cdots w_{n-1} \sqcup v)w_n$.

$$v \bar{\sqcup} w := v \sqcup w[\max(v)], \text{ where } w[k] = (w_1 + k)(w_2 + k) \cdots (w_n + k).$$

If $\sigma = 312$ and $\tau = 12$:

$$\begin{aligned} \mathbf{F}_{312} \mathbf{F}_{12} &= \mathbf{F}_{31245} + \mathbf{F}_{31425} + \mathbf{F}_{34125} + \mathbf{F}_{43125} + \mathbf{F}_{31452} \\ &\quad + \mathbf{F}_{34152} + \mathbf{F}_{43152} + \mathbf{F}_{34512} + \mathbf{F}_{43512} + \mathbf{F}_{45312}. \end{aligned}$$

Dendriform structures **FQSym**

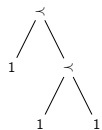
$$\mathbf{F}_\sigma \prec \mathbf{F}_\tau := \sum_{\substack{\gamma \in \sigma \bar{\sqcup} \tau \\ \gamma_{n+m} = \sigma_n}} \mathbf{F}_\gamma,$$

and

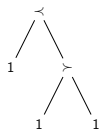
$$\mathbf{F}_\sigma \succ \mathbf{F}_\tau := \sum_{\substack{\gamma \in \sigma \bar{\sqcup} \tau \\ \gamma_{n+m} = \tau_m[n]}} \mathbf{F}_\gamma.$$

Remark : $\bar{\sqcup} = \prec + \succ$.

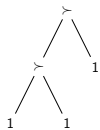
Dendriform algebras



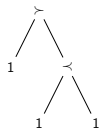
321



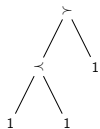
231



123



132 + 312



213

$$\dim(\mathcal{A}_n) \geq C_n.$$

Dendriform algebras

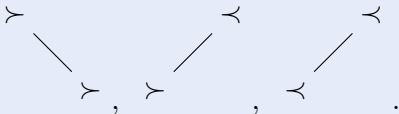
Theorem (Loday 1998)

Hilbert series of \mathcal{A} , the free dendriform algebra over one generator of degree one :

$$\sum_{n \geq 1} C_n t^n.$$

Theorem (Novelli 2014)

A basis of \mathcal{A} : the family \mathcal{F} of complete binary trees labeled by $\{\gamma, \prec\}$ and avoiding



Dendriform algebras

Theorem (Foissy 2007)

FQSym is a free dendriform algebra.

Theorem (V. 2014)

Freeness of **FQSym** \approx bijection between permutations and a family of trees.

Corollary

If $\mathcal{H} = \sum_{n \geq 1} n! t^n$ and $\mathcal{G} = \sum_{n \geq 1} C_n t^n$, then :

$$\mathcal{G}^{-1} \circ \mathcal{H} = \sum_{n \geq 1} a_n t^n,$$

with $a_n \in \mathbb{N}$.

Thanks for your attention.