An instance of analysis of algorithms: The plain GCD algorithm

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Based on a joint work
with Valérie Berthé, Jean Creusefond and Loïck Lhote.

Journées nationales du GDR Informatique Mathématique
Bordeaux, Février 2015.

Computing the gcd of ℓ inputs

For $\ell = 2$: the "classical" Euclid algorithm :

a sequence of Euclidean divisions

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but not yet analyzed for $\ell > 2$ (Problem HM 48)

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The same phenomena occur for the size of the partial gcd.

Plan of the talk.

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I. Probabilistic analysis of algorithms.

Based on Analytic Combinatorics and Generating Functions.

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Based on Dynamical Combinatorics and Dirichlet Generating Functions

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IV. An unified point of view for the two analyses?

A dynamical point of view.

I. Probabilistic analysis of algorithms

Based on Analytic Combinatorics
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$$\mathbb{E}_n[L] \underset{n\to\infty}{\sim} a_n, \quad \mathbb{V}_n[L] \underset{n\to\infty}{\sim} b_n, \quad \mathbb{P}_n\left[\frac{L - a_n}{\sqrt{b_n}} \in [x, x + dx]\right] \underset{n\to\infty}{\sim} f(x)dx$$

Examples of limit laws (discrete or continuous)

x-axis: possible values of the cost $L(\omega)$

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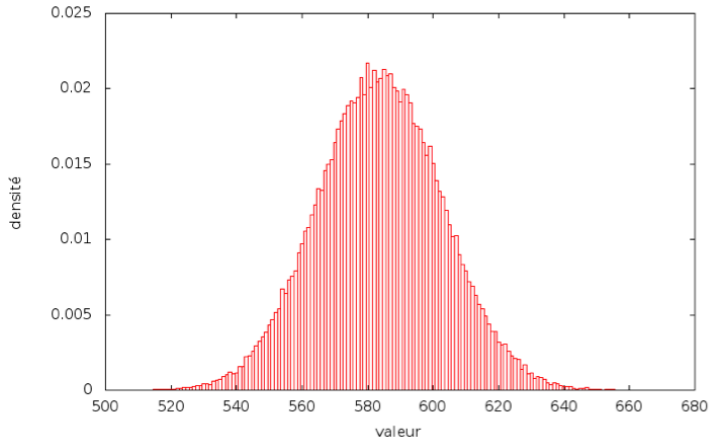
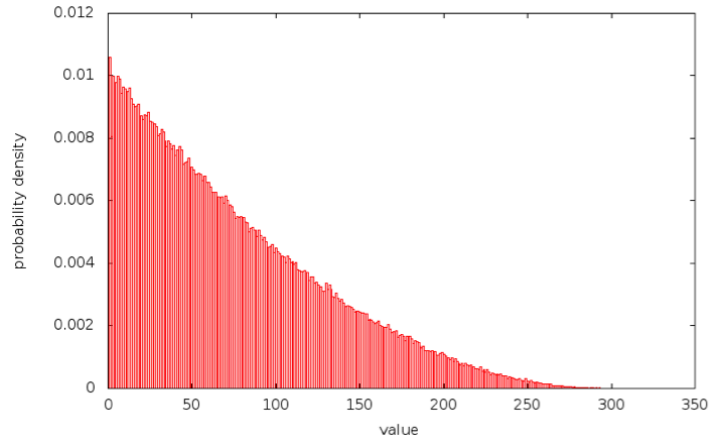
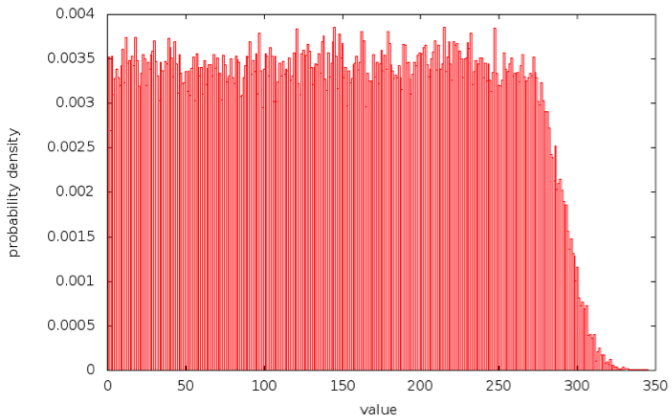


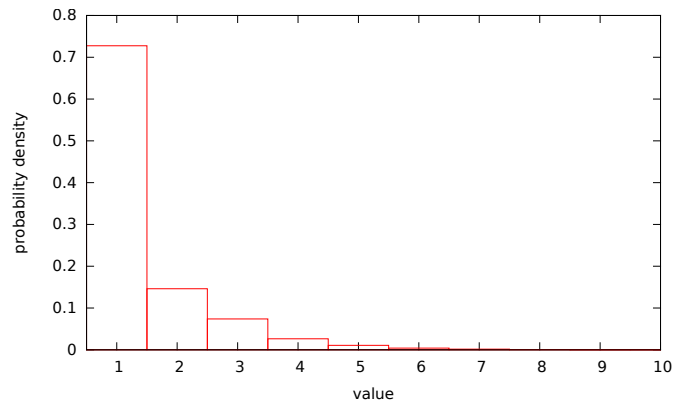
Gaussian law $f(t) \asymp e^{-t^2/2}$



Beta law (a, b) $f(t) \asymp t^{a-1}(1-t)^{b-1}$



Uniform law $f(t) \asymp 1$



A discrete law: Geometric law $f(n) \asymp a^n$

Usual analysis for the Euclid Algorithm $\ell = 2$.

The usual size of $\underline{x} = (x_1, x_2)$ is the maximum of the sizes of the inputs:

$$\mathrm{d}(\underline{x}) := \max(\mathrm{d}(x_1), \mathrm{d}(x_2)).$$

The two costs of interest are :

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Its distribution is asymptotically geometric.

This (usual) size is not adapted for a number of inputs $\ell > 2$.

For $\ell \geq 2$, we consider the total size and this choice modifies the results

$\Longrightarrow$ Irruption of the (unexpected) beta law

Analysis of algorithms and generating functions.

Two main cases here, depending on

– the type of the input $\underline{x} = (x_1, x_2, \ldots, x_\ell)$, $x_i \in \mathbb{N}$ or $x_i \in \mathbb{F}_q[X]$

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– for polynomials: the total degree $\mathrm{d}(\underline{x}) = \mathrm{d}(\prod x_i) = \sum \mathrm{d}(x_i)$

$\Omega_n := \{\underline{x} = (x_1, x_2, \ldots, x_\ell); \quad \mathrm{d}(\prod x_i) = n\}$

Power generating functions for polynomials

$$S(z) := \sum_{\underline{x} \in \Omega} z^{\mathrm{d}(\prod x_i)} \quad \Longrightarrow \quad |\Omega_n| = [z^n] S(z)$$

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– for integers: the total length $\quad \mathrm{d}(\underline{x}) = \lfloor \log \prod x_i \rfloor \sim \sum \lfloor \log x_i \rfloor$

$$\Omega_n := \{\underline{x} = (x_1, x_2, \ldots, x_\ell); \quad e^n \leq \prod x_i < e^{n+1}\}$$

Dirichlet generating functions for integers

$$S(s) := \sum_{\underline{x} \in \Omega} (\prod x_i)^{-s} \quad \Longrightarrow \quad |\Omega_n| = \sum_{e^n \leq N < e^{n+1}} [N^{-s}] S(s)$$

Analysis of algorithms and generating functions (II).

$$\mathbb{P}_n[L=m]=\frac{|\Omega_{n,m}|}{|\Omega_n|} \qquad \text{with} \quad \Omega_{n,m}=\{\underline{x}\in\Omega_n;\quad L(\underline{x})=m\}$$

Numerators are expressed with coefficients of bivariate generating functions.

For polynomials:

$$\Omega_{n,m}:=\{\underline{x}=(x_1,x_2,\ldots,x_\ell\};\quad \mathrm{d}(\textstyle\prod x_i)=n,\quad L(\underline{x})=m\}$$

$$S(z):=\sum_{\underline{x}\in\Omega} z^{\mathrm{d}(\prod x_i)}, \qquad S(z,u):=\sum_{\underline{x}\in\Omega} z^{\mathrm{d}(\prod x_i)}u^{L(\underline{x})}$$

$$\mathbb{P}_n[L=m]=\frac{[z^nu^m]S(z,u)}{[z^n]S(z)}$$

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Analysis of algorithms and generating functions (GF): main principles.

Distributional analysis is related

– to the asymptotic behaviour of coefficients of the BGF's

– obtained with two steps:

Combinatorial step.

Translate the structure of the algorithm on the generating functions:

- obtain an algorithmic expression of the generating function,
- from which the dominant singularity becomes apparent.

Analytic step.

For any GF, the asymptotic behaviour of coefficients is related to

- the position and the nature of its dominant singularity
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The same principles in the two analyses and two main differences.

– Dirichlet GF's more difficult to study than power GF's

– For integers, there are correlations due to carries.

II. Analysis of the polynomial case.

Based on Analytic Combinatorics
and Generating Functions

Probabilistic analysis of the plain ℓ–GCD algorithm on $\mathbb{F}_q[X]$.

On the input $(x_1, x_2, \ldots, x_\ell)$,

- the algorithm computes the total gcd $y_\ell := \gcd(x_1, x_2, \ldots, x_\ell)$
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The set of inputs is $\Omega = \{(x_1, \ldots, x_\ell); \quad x_i \ \text{ monic } \ \in \mathbb{F}_q[X]\}$

The size of an input : $\mathrm{d}(x_1, \ldots, x_\ell) = \mathrm{d}(x_1) + \ldots + \mathrm{d}(x_\ell)$,

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Main costs of interest

- the number L_k of divisions during the k–th phase

i.e. on the input (x_k, y_{k-1})

- the degree D_k of the k–th gcd

(at the beginning of the k-th phase).

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The pair (a_1, a_2) of monic polynomials is entirely determined by

– the sequence of quotients $(m_1, m_2, \dots, m_r)$,

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$\Omega = \mathcal{U}^2$	$\approx$	{first quotient}	$\times$	{sequence of quotients}	$\times$	{GCD}
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Algorithmic expression of the GF of the ℓ-Euclid Algorithm

We have shown that the Euclid algorithm ($\ell = 2$) translates as a product

$$U(z_1)\,U(z_2) = T(z_1, z_2)\;U(z_1 z_2), \qquad \text{with} \quad T(z_1, z_2) = \frac{U(z_1) + U(z_2) - 1}{1 - G(z_1 z_2)}$$

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$$U(z_1) \cdot \ldots \cdot U(z_\ell) = U(t_\ell) \prod_{k=1}^{\ell-1} T(z_{k+1}, t_k) \qquad [t_k := z_1 \cdot z_2 \cdot \ldots \cdot z_k]$$

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Now, with $z = z_1 = \ldots = z_\ell$,
the (plain) generating function $S(z)$ of $\mathcal{U}^\ell$ has the alternative expression

$$S(z) = U(z)^\ell = U(z^\ell) \prod_{k=1}^{\ell-1} T(z, z^k)$$

which is an exact translation of the ℓ-Euclid algorithm.

T is the "phase generating function".

Bivariate Generating Functions relative to the ℓ-Euclid Algorithm

We start with: $$S(z) = U(z)^{\ell} = U(z^{\ell}) \prod_{k=1}^{\ell-1} T(z, z^k)$$

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- L_k (number of steps in the k-th phase)
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$$L_k(z,u) = U(z)^\ell \cdot \frac{T(z, z^k, u)}{T(z, z^k)}, \qquad D_k(z,u) = U(z)^\ell \cdot \frac{U(z^k, u)}{U(z^k)},$$

With:

$$T(z,t,u) = u\frac{U(z) + U(t) - 1}{1 - uG(zt)}, \qquad U(t,u) = \frac{1}{1 - qut}.$$

Towards the distributional analysis of L_k and D_k.

$$\mathbb{P}_n[L_k > m] = \sum_{j>m} \frac{[u^j z^n] L_k(z,u)}{[z^n]S(z)} = \frac{[z^n]\sum_{j>m}[u^j]L_k(z,u)}{[z^n]S(z)} = \frac{[z^n]\widehat{L}_k^{[m]}(z)}{[z^n]S(z)}$$

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The generating functions "of the numerators" are

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The asymptotics depends on the value $a := A_k(1/q)$ at the pole $z = 1/q$

- For the first phase $k = 1$, one has $a = 1$
- For the subsequent phases $k \geq 2$, one has $a < 1$

A general result for the asymptotics of coefficients

Consider the function $F^{[m]}(z) = \dfrac{1}{(1-z)^{\ell}}\, A(z)^m, \quad$ with $\ell \geq 2$

where (i) $A(z)$ is analytic on the disk $|z| \leq \rho$ with $\rho > 1$,

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(iii) for $|z|$ close enough to 1, $|A(z)| \leq A(|z|)$.

Then, for $n \to \infty$ and $m/n \in [0, a/b[$,

$$[z^n]F^{[m]}(z) = \frac{n^{\ell-1}}{(\ell-1)!}\, a^m \left(1 - \frac{b}{a}\frac{m}{n}\right)^{\ell-1} \left[1 + O\left(\frac{1}{n}\right)\right].$$

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Application to the present situation.

For the first phase: $a = 1 \implies$ A "beta" behavior $(1, \ell - 1)$

For the subsequent phases: $a < 1 \implies$ A geometric behavior of ratio a

Main result for the number of divisions L_k – First phase $(k = 1)$

The number of divisions L_1 performed during the first phase

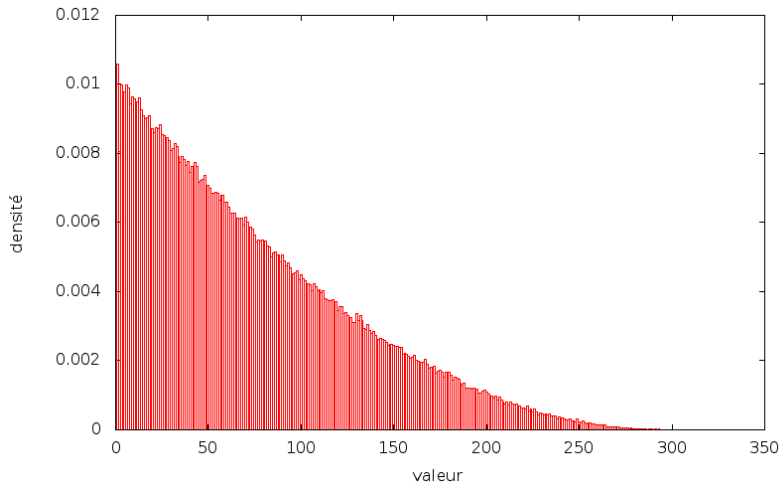
– has a mean value of linear order

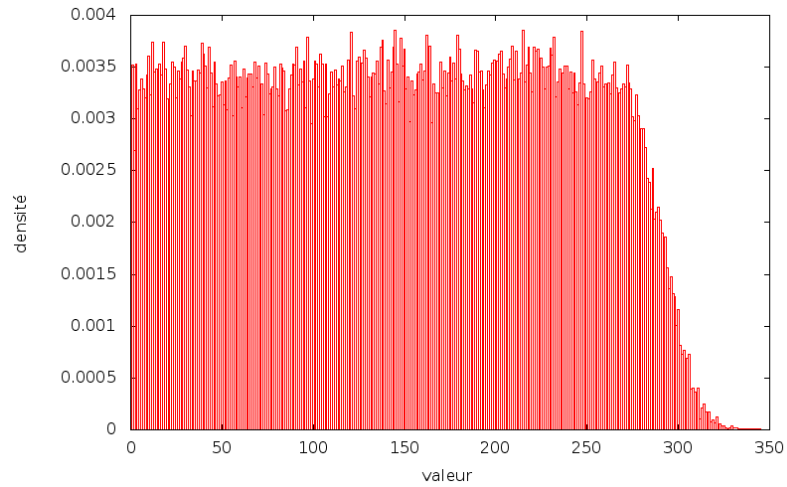
$$\mathbb{E}_n[L_1] = \frac{q-1}{2q}\frac{n}{\ell} + \frac{3q+1}{4q} + O\left(\frac{1}{n}\right). \qquad \frac{2q}{q-1} = \text{entropy}$$

– follows an asymptotic beta law of parameter $(1, \ell - 1)$.

Its distribution satisfies when $n \to \infty$, and $m/n \in [0, (q-1)/(2q)]$

$$\mathbb{P}_n[L_1 > m] = \left(1 - \frac{2q}{q-1}\frac{m}{n}\right)^{\ell-1} + O\left(\frac{1}{n^\alpha}\right).$$





Main result for the number of divisions L_k – Subsequent phases (case $k \geq 2$)

For $k \geq 2$, the number of divisions performed during the k-th phase

– has a mean value of constant order

$$\mathbb{E}_n[L_k] = \left(1 + \frac{q-1}{q^k - q}\right) + O\left(\frac{1}{n}\right),$$

– follows an asymptotic geometric law, with ratio $\dfrac{q-1}{q^k-1}$

For $n \to \infty$ and $m/n \in [0, 1/(k+1) \cdot (q^k-1)/q^k]$

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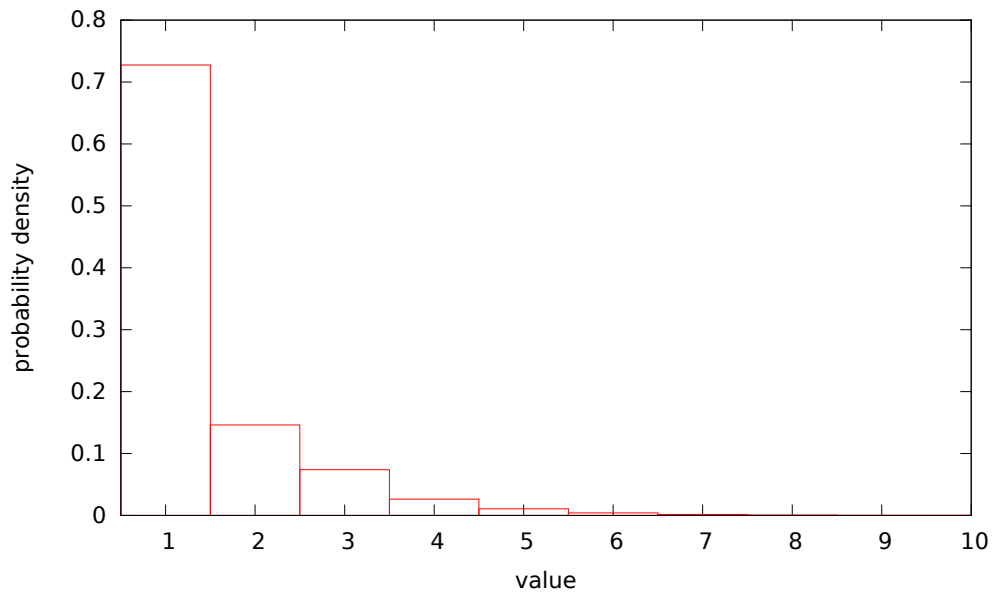

III. Analysis on integers.

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We use Dirichlet generating functions
together with a dynamical point of view

Remind : The Euclid algorithm ($\ell = 2$) on polynomials

translates as a product of power generating functions

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An instance of a "dynamical" analysis....

More involved than the previous one, but provides the same type of results.

Similarities and differences between the two analyses

	Polynomials	Integers
GF	Power GF's	Dirichlet GF and operators
Basic tool	$G(z) = \sum_m z^{\mathrm{d}(m)}$	$\mathbf{G}_s[f](t) = \sum_{m \geq 1} \left(\frac{1}{m+t}\right)^s f\left(\frac{1}{m+t}\right)$
Phase GF	$\frac{U(z^k) + U(z) - 1}{1 - G(z^{k+1})}$	$(I - \mathbf{G}_{(k+1)s})^{-1} \circ (\mathbf{G}_{ks} + \mathbf{G}_s)[1](0)$
Singularities	z s.t. $G(z^{k+1}) = 1$	s s.t. $\lambda((k+1)s) = 1$
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$\lambda(s)$ is the dominant eigenvalue of $\mathbf{G}_s$

$\lambda(2)=1$; $\lambda'(2)$ closely related to the entropy

Results in the integer case.

NB: the integer size of the input $\approx$ number of digits in base e

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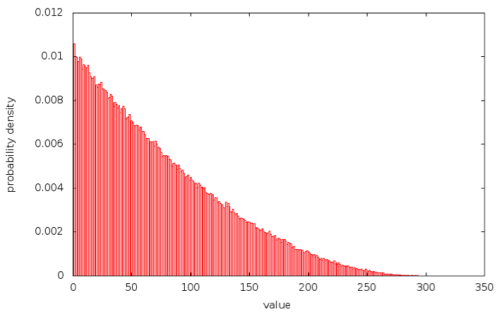
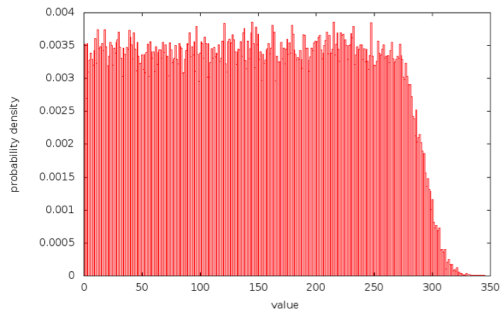
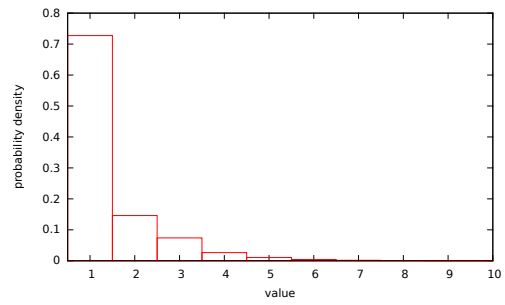
The same phenomena occur for the size of the partial gcd.







Probabilistic analysis of the plain ℓ–GCD algorithm on $\mathbb{Z}$.

On the input $(x_1, x_2, \ldots, x_\ell)$,

- the algorithm computes the total gcd $y_\ell := \gcd(x_1, x_2, \ldots, x_\ell)$
- with $\ell - 1$ phases.
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Main costs of interest

- the number L_k of divisions during the k–th phase

i.e. on the input (x_k, y_{k-1})

- the size D_k of the k–th gcd

(at the beginning of the k-th phase).

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For LFT's, the denominators are recovered with the derivatives

$$\text{if} \quad h(x) = \frac{ax+b}{cx+d}, \quad \text{then} \quad h'(x) = \frac{\det h}{(cx+d)^2}$$

Proof of the alternative expression for the Dirichlet generating function ($\ell = 2$)

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Using LFT's as in the previous slide :

$$\sum_{\substack{(u_1,u_2),u_2\leq u_1\\ \gcd(u_1,u_2)=1}} \frac{1}{u_1^{s_1}}\frac{1}{u_2^{s_2}} = \sum_{\substack{g\in\mathcal{H},\\ h\in\mathcal{H}^\star}} |(g\circ h)'(0)|^{s_1/2}|h'(0)|^{s_2/2}$$

$$= \sum_{\substack{g\in\mathcal{H},\\ h\in\mathcal{H}^\star}} |g'(h(0))|^{s_1/2}|h'(0)|^{(s_1+s_2)/2}$$

Proof of the alternative expression for the Dirichlet generating function ($\ell = 2$)

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1+s_2)\left(\frac{1}{2}\,(I - \mathbf{G}_{s_1+s_2})^{-1} \circ (\mathbf{G}_{s_1} + \mathbf{G}_{s_2})\right)[1](0)$$

with $\quad \mathbf{G}_s[f](t) = \sum_{h\in\mathcal{H}} |h'(t)|^{s/2} f\circ h(t) = \sum_{m\geq 1} \left(\frac{1}{m+t}\right)^s f\left(\frac{1}{m+t}\right)$

Proof. The Dirichlet generating function of the inputs is

$$\sum_{(a_1,a_2)\in\mathbb{N}^2} \frac{1}{a_1^{s_1}}\frac{1}{a_2^{s_2}} = \zeta(s_1)\cdot\zeta(s_2), \qquad \zeta(s) = \sum_{a\in\mathbb{N}} \frac{1}{a^s}$$

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The Euclid algorithm ($\ell = 2$) on integers

translates as a product of Dirichlet generating functions

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$$\zeta(s_1) \cdot \ldots \cdot \zeta(s_\ell) = \zeta(t_\ell) \prod_{k=1}^{\ell-1} T(s_{k+1}, t_k) \qquad [t_k := s_1 + s_2 + \ldots + s_k,]$$

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Now, with $s = s_1 = \ldots = s_\ell$,
the (plain) generating function $S(s)$ of $\mathbb{N}^\ell$ has an alternative expression

$$S(s) = \zeta(s)^\ell = \zeta(\ell s) \prod_{k=1}^{\ell-1} T(s, ks)$$

which is an exact translation of the ℓ-Euclid algorithm.

T is the "phase generating function".

Bivariate Generating functions for the ℓ-Euclid Algorithm

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$$L_k(s,u) = \zeta(s)^\ell \cdot \frac{T(s,ks,u)}{T(s,ks)}, \qquad D_k(s,u) = \zeta(s)^\ell \cdot \frac{Z(ks,u)}{\zeta(ks)},$$

$$T(s,t,u) = \frac{1}{2} u (1 - u\mathbf{G}_{s+t})^{-1} \circ (\mathbf{G}_s + \mathbf{G}_t)[1](0), \qquad Z(s,u) = \sum_{n \geq 1} \frac{u^{\mathrm{d}(n)}}{n^s}$$

Towards the distributional analysis of L_k and D_k.

The generating functions of the events $[L_k > m]$ and $[D_k > m]$

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The asymptotics depends on the value $a := A_k(1)$ at the pole $s = 1$ of $B(s)$

For $k = 1$, one has $a = 1$ – For $k \geq 2$, one has $a < 1$.

Main result for the number of divisions L_k – First phase ($k = 1$)

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The number of divisions L_1 performed during the first phase

– has a mean value of linear order

$$\mathbb{E}_n[L_1] = \frac{6\log 2}{\pi^2}\,\frac{n}{\ell}\left[1 + O\left(\frac{1}{n}\right)\right] \qquad \frac{\pi^2}{6\log 2} = \text{entropy}$$

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Its distribution satisfies when $n \to \infty$, and $m/n \in [0, (6\log 2)/\pi^2]$

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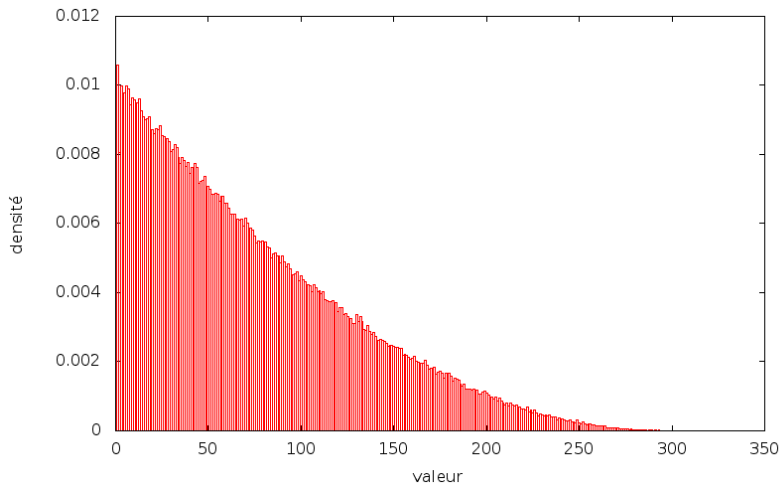

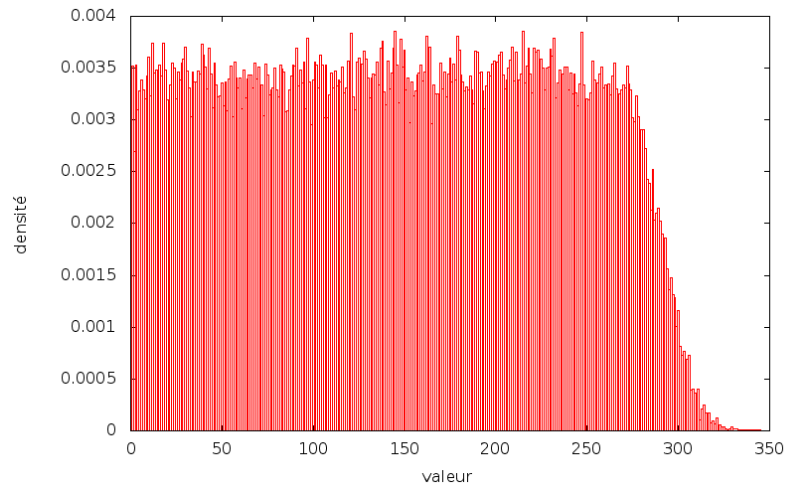

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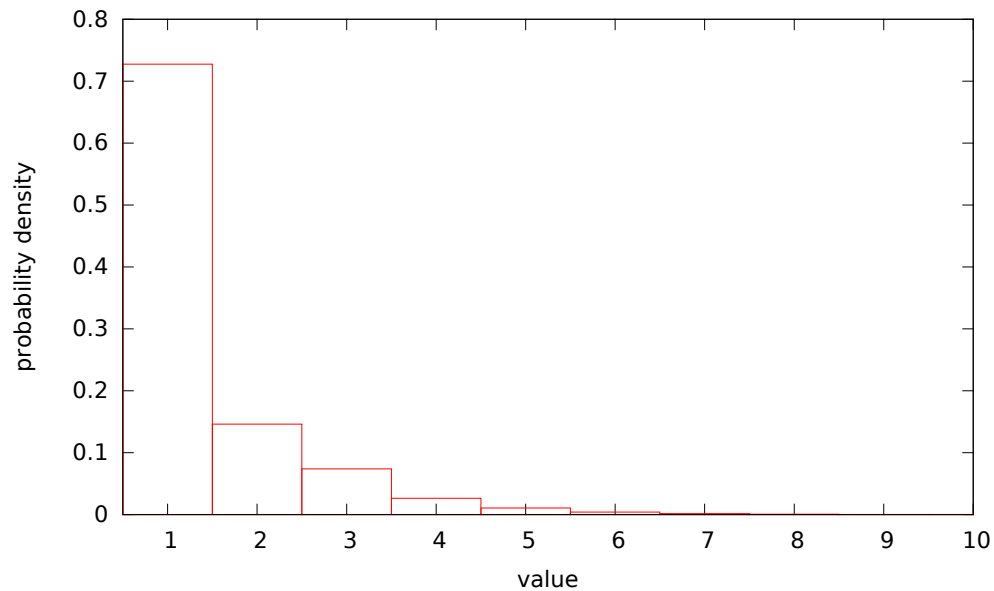


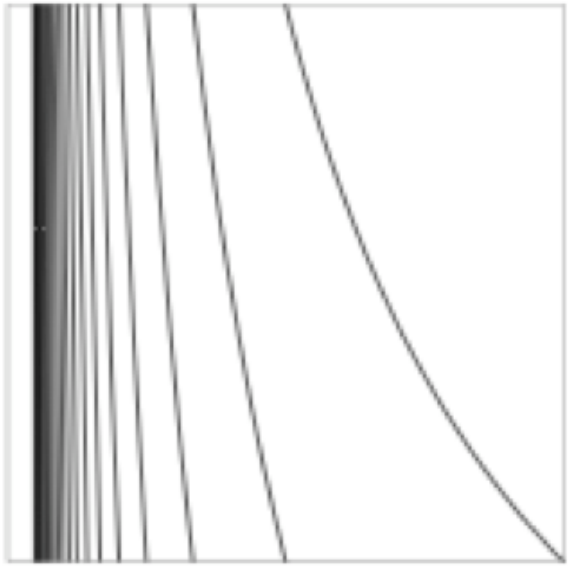
IV. An unified point of view for the two analyses?

A dynamical point of view....

The underlying dynamical system in the integer case

The underlying dynamical system in the integer case defined by the pair $([0,1], V)$.

$$V : [0,1] \longrightarrow [0,1], \quad V(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for} \quad x \neq 0, \quad V(0) = 0$$



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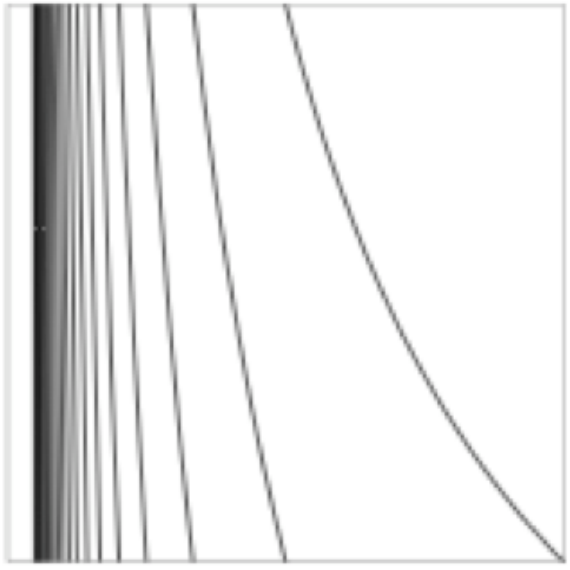

The set of the inverse branches of V is

$$\left\{ h_m(x) = \frac{1}{m+x} \right\}$$

It builds the continued fraction expansions.

The operator $\mathbf{G}_s$ is the transfer operator of the dynamical system.
It involves the derivatives of the inverse branches of V

$$\mathbf{G}_s[f](x) = \sum_{m \geq 1} \left(\frac{1}{m+x} \right)^s f\left(\frac{1}{m+x} \right)$$

The underlying dynamical system in the polynomial case.

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The exact ultrametric analog of the DS in the integer case.

$$\mathbb{F}_q[Z] \longleftrightarrow \mathbb{Z}, \qquad \mathbb{F}_q(Z) \longleftrightarrow \mathbb{Q}, \qquad \mathbb{F}_q((1/Z)) \longleftrightarrow \mathbb{R}$$

$\mathbb{F}_q((1/Z))$: the completion of $\mathbb{F}_q[Z]$ for the ultrametric norm $||u|| := q^{\mathrm{d}(u)}$.

The integer part function $\lfloor \cdot \rfloor$ satisfies $\lfloor f \rfloor = 0$ iff $||f|| \leq 1$

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Classical analysis for polynomials: a particular case of a dynamical analysis

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