

Stable sets in Graphs

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- $\omega(G) \leq \chi(G)$. (**Gap may be arbitrary even when $\omega = 2$.**)
- G is **perfect** when all induced subgraphs H satisfy

$$\omega(H) = \chi(H)$$

Main question

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Question

Stable sets should behave well (or better...) in strict classes. How to express "good behaviour"?

Computing α

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- Berge graphs. (**See later**)
- Line graphs. (**Edmonds**)

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Uses **potential maximal cliques** (Bouchitté and Todinca).

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- G is Berge (Chudnovsky, Robertson, Seymour, Thomas).

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Theorem (Lovász)

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Theorem (Rothvoss, STOC 2014)

STAB of line graphs does not have extended formulations.

χ -bounded classes

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Theorem (Scott and Seymour 2014)

Graphs not inducing odd cycles of length at least 5 are χ -bounded.

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Wanted

Every graph with large chromatic number has a "rather complex" induced subgraph.

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Kalai and Meshulam conjecture that if G has no induced $0 \pmod 3$ cycle, the number of odd stable sets and the number of even stable sets differ by at most 1.

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Theorem (Bousquet, Lagoutte and T., 2013)

For every k , the Erdős-Hajnal conjecture holds for $(P_k, \overline{P_k})$ -free graphs.

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Proposition (Alon et al.)

Linear bipartite property implies Erdős-Hajnal.

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For every k and ε , there exists c such that every graph on n vertices contains one of the following:

- all induced graphs on k vertices.
- a subset of size $c.n$ with edge density at least $1 - \varepsilon$.
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Remark

Easy proof with bad bounds via Szemerédi regularity lemma, harder proof with good bounds provided by Fox and Sudakov

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End of the proof, à la Gyárfás

Complexity of graph classes?

Some guesses

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- Propose your own...